The Characterizations of $\in$-d-Chainable Sets in 2-Metric Spaces

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Abstract

In the present paper, we introduce the concept of $\in$-d-chain, $\in$-d-chainable sets and uniformly chainable sets with examples in 2-Metric space and prove the relation between $\in$-d-chainable points and $\in$-d chainable sets and at last an equivalence relation is investigated.
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Introduction

Connectedness and compactness are widely studied in Topology. In 1883 Cantor defined connectedness with the help of $\in$-chains. At present however the Riesz Hausdorff definition using the idea of separated sets, is universally accepted. On the other hand a lot of experiments have led to several forms of compactness. Compactness and several of its generalizations are defined in terms of open covers e.g. compact, countably compact, paracompact, Lindeoff etc. Chainablity characterizes connected sets among compact sets in setting of metric spaces. After a long time Shrivastava and Agrawal (2002) defined ε -chainable sets in metric space. It has been proved that chainability of points and sets is product invariant property (Pagey *et al*., 2010). Before that proved many fixed point theorems in ε-chainable metric spaces (Pagey *et al*., 2009). Because of its importance this topic has attracted attention of mathematicians. Agrawal *et al.* (2015) defined locally chainable sets in metric spaces and proved various related results. Very recently Agrawal *et al.* (2017) investigated relation between chainable and compact sets defined uniformly chainable sets, self- chainable sets, strong chainable sets in metric spaces and proved various related results with application in Fixed point theory.

In this paper, we define an $\in$-d chain between three points in 2-metric space. We extend the concept of $\in$-d-chain between three points to $\in$-d-chain between two sets in 2-metric spaces. This generalization yields a simple characterization of $\in$-d-chainable sets in terms of $\in$-d-chains between three points and it is shown that d-chainability between sets, implies d-chainability between their closures and lastly results on compact sets relating to. $\in$-d-chainability and equivalence relation
have been established .

2. Preliminaries

Definition 2.1: [4] Gahler defined a 2- metric space as follows. Let d: $X \times X \times X \to R^+$ with the following properties:

(1) For distinct points $x,y \in X$ there exists a point $z \in X$ such that $d(x,y,z) \neq 0$
(2) $d(x,y,z) = 0$ if at least two of x,y,z are equal
(3) $d(x,y,z)=d(P\{x,y,z\})$ where P is permutation of x,y,z (Symmetry)
(4) $d(x,y,z) \leq d(x,y,a) + d(x,a,z) + d(a,y,z)$ $\forall$ $x,y,z,a \in X$ (Tetrahedral inequality)

If d is a 2-metric on X then the pair (X,d) is called a 2-metric space. Throughout this paper X will stand for 2-metric space with 2-metric d.

Example 2.1 (a): Let X be an arbitrary non empty set we define d by

$$d(x,y,z) = \begin{cases} 0 & \text{if any two of the triplet } x, y, z \text{ are equal where } x,y,z \in X \\ 1 & \text{if } x \neq y \neq z \quad \forall\, x,y,z \in X \end{cases}$$

then (X,d) is a 2-metric space.

Example2.1 (b):[6] Let X be a non empty set and the function $d: X\times X\times X \to R^{+}$ defined by

$d(x,y,z) = \text{Min}\{\rho(x,y), \rho(y,z), \rho(z,x)\}$ Where ρ is the usual metric on X then (X,d) is a 2- metric space.

Definition 2.2: A sequence $\{x_n\}$ in a 2- metric space X is said to be

(1) convergent with the limit x in X $\lim\limits_{n\to\infty} d(x_n,x,a) = 0 \ \forall\, a\in X$.

Definition 2.3:[3], [5] let (Y, ρ) be a metric space For points p, q $\in$ Y an *ε-chain* of length n from p to q is a finite sequence $a_0, a_1, a_2, \ldots\ldots\ldots\, a_n$ in X with $a_0=p$, $a_n = q$ and $d(a_{i-1}, a_i) < \varepsilon$, $1 \le i \le n$.

We call Y ϵ-chainable if each two points in Y can be joined by an ϵ-chain and Y is chainable if Y is ϵ-chainable for each positive ϵ.

3 Main Definitions

Definition 3.1 : For points p, q, r $\in$X an ϵ-d-chain of length n from p to q and r is a finite sequence

$$a_0 \begin{array}{l} a_1, a_2, \ldots\ldots\ldots\, a_n \\ \acute{a}_1, \acute{a}_2, \text{-- - ---------}\, \acute{a}_n \end{array}$$

in X

where $a_0 = p$ and $a_n = q$, $\acute{a}_n = r$ and $d(a_{i-1}, a_i, \acute{a}_i) < \varepsilon$, $1\le i \le n$

We call X ϵ-d-chainable if each three points in X can be joined by an ϵ-d-chain and X is d-chainable if X is ϵ-d-chainable for each positive ϵ.

Definition 3.2: For any set A in a 2--metric space X the d-diameter of A denoted by δ(A) is defined by

$$\delta(A) = \sup_{x,y,z\in A} d(x,y,z)$$

Definition 3.3: Let A be a sub set of the 2-metric space (X,d) For ϵ>0

Let $V\in(A) = \{y,z \in X : d(A,y,z) < \epsilon\}$ Where $d(A,y,z) = \inf\{\mathbf{d}(x,y,z) : x\in A\}$

Definition 3.4: Let A,B $\subset$X an $\in$-d-chain of length n from A to B is a finite sequence $A_0, A_1, \text{------}\, A_n$ of subsets of X with $A = A_0$, $A_n = B$ $A_{i-1}\subset V\in(A_i)$ and $A_i\subset V\in(A_{i-1})$ $1\le i \le n$ if

$\in$-d-chain exists between A and B we say $< A,B >$ is $\in$-d-chainable and A,B is d-chainable if it is $\in$-d-chainable for each positive $\in$.

Inductivity construct the sets $V\in^{n}(A)$ for each $n\in Z^{+}$ as follows $V\in^{1}(A) = V\in(A)$ and for $n \geq 2$ set $V\in^{n}(A) = V\in (V\in^{n-1}(A))$. The following should be observed

(1) $\quad V\in^{n}(A) \subset V\in^{n+1}(A)$

We set $\Phi_{\epsilon}(< A,B >)$ to be the length of shortest ϵ-d-chain between A and B.

Example 3.5(a): Let $< x_n >$ be a real sequence such that x_n is convergent to the point x Let

$A = \{ x_n : n \in N, n \geq n_0 \}$ where n_0 is any fixed natural number and $B=\{ x \}$

then $A \subset V\in(B)$ and $B \subset V\in(A)$

[by definition 2.2] $\Rightarrow < A,B > \in$-d-chainable and $\Phi_{\epsilon}(< A,B >) = 1$

Example3.5(b): Let $Y = A \cup B$ and ρ be the usual metric on Y. The 2-metric d defined by

$$d(x,y,z) = \text{Min}\{ \rho(x,y), \rho(y,z), \rho(x,z)\}$$

If $A = \{ 1, 1/2, 1/3, \text{------------}\}$ and $B = \{ 1/2^2, 1/3^2, 1/4^2 \text{--------}\}$

Then $< A,B >$ is $\in$-d-chainable and $\Phi_{\epsilon}(< A,B >) =1$ for $\epsilon = 1$

Example 3.5(c): Consider the 2-metric space X where d is the 2-metric and defined by

$$d(x,y,z) = \begin{cases} 0 & \text{if any two of the triplet } x, y, z \text{ are equal where } x,y,z \in X \\ 1 & \text{if } x \neq y \neq z \quad \forall\, x,y,z \in X \end{cases}$$

Then for any two subsets A,B of X $< A,B >$ is $\in$-D-chainable for $\in > 1$ and $\Phi_{\epsilon}(< A,B >) = 1$

Definition 3.6 Let A,B is sub set of X ,$< A,B >$ is called uniformly $\in$-d-chainable if there exist a positive integer n such that each point of A can be joined two point of B by an $\in$-d-chain of length at most n and vice versa also $< A,B >$ is called uniformly d-chainable if it is uniformly $\in$-d-chainable.for all positive $\in$

Definition 3.7: Let A be a subset of X then A is said to be self $\in$-d-chainable if every three points of A can be joined by an $\in$-d-chain. Also A is said to be self d-chainable if A is self$\in$-d-chainable for every $\in > 0$.

Some Related Results

I . Let $A,B \in X$, then

(1) $\quad A \subset V\in(A)$

(2) $\quad A \subset B \Rightarrow V\in(A) \subset V\in(B)$

(3) $\quad V\in(A) \cup V\in(B) = V\in(A \cup B)$

(4) $\quad V\in(A \cap B) \subseteq V\in(A) \cap V\in(B)$

(5) $A \subseteq \bigcap_{\epsilon>0} V_\in(A) = \bar{A}$

(6) A is closed iff $\bar{A} = \bigcap_{\epsilon>0} V_\in(A)$

II . If $< A,B >$ and $<C,D>$ are d-chainable then $< A \cup C ,B \cup D >$ is also d-chainable where A,B,C,D is the subsets of X

4. MAIN RESULT

Theorem 4.1: Let $A,B \subset X$ and $< A,B >$ be $\in$-d-chainable then there exists an $\in$-d-chain from every point of A to some two point of B and vice versa also the converts holds.

Proof: We prove the necessary part first As $< A,B >$ is $\in$-d-chainable there exists a sequence

$A_0, A_1 \text{------} A_n$ of subsets of X with $A = A_0$, $A_n = B$ $A_{i-1} \subset V_\in(A_i)$ and $A_i \subset V_\in(A_{i-1})$ $1 \leq i \leq n$

Let $x \in A$ be arbitrary The $x \in A \Rightarrow x \in V_\in(A_1) \Rightarrow d(x, x_1, x_1^{/}) < \in$ for some $x_1, x_1^{/} \in A_1$

Again $x_1 \in A_1 \Rightarrow d(x_1, x_2, x_2^{/}) < \in$ for some $x_2, x_2^{/} \in A_2$ Repeating the above process n times
We obtain a sequence of points

$$
\begin{array}{llll}
 & x_1 \quad x_2 \quad x_3 \text{-------------} x_n = y \in B \\
x_0 & \\
 & x_1^{/} \quad x_2^{/} \quad x_3^{/} \text{-------------} x_n^{/} = z \in B
\end{array}
$$

Such that $d(x_{i-1}, x_i, x_i^{/}) < \in \quad 1 \leq i \leq n$

and $x_i , x_i^{/} \in A_i$ showing that there exists an $\in$-d-chain from x to y , z likewise. We can obtain an $\in$-d-chain from every point of B to some two point of A.

We next prove the sufficient part Let there exist an $\in$-d-chain from every point of A to some two point of B and vice versa.

$x \quad x_1 \quad x_2 \quad x_3 \text{-------------} x_n = y \in B$ $A_1 = \{ a,b \in X : d(a, b, x) < \in$ for some $x \in A$ and $a \neq x$ and $b \neq x \}$

Clearly $A_1 \neq \phi$ and $A_1 \subset V_\in(A)$

Next we show that $A \subset V_\in(A_1)$. If $x, x^{/} \in A$ then there exist a sequences

$$x^{/} \quad x_1^{/} \quad x_2^{/} \quad x_3^{/} \text{-------------} x_n^{/} = z \in B$$

such that $d(x , x_1 , x_1^{/}) < \in \Rightarrow x_1 , x_1^{/} \in A_1$ then as $A_1 \subset V_\in(A)$

$\Rightarrow \quad d(x_1 , x , x^{/}) < \in \Rightarrow d (A_1 ,x , x^{/}) < \in \Rightarrow x , x^{/} \in V_\in(A_1)$

again Let$A_2 = \{ a,b \in X : d(a, b, x) < \in$ for some $x \in A_1$ and $a \neq x$ and $b \neq x \}$

Clearly $A_2 \neq \phi$ and $A_2 \subset V_\in(A_1)$ and it can be shown as above that $A_1 \subset V_\in(A_2)$ Repeating the above process n times we obtain a sequence $A = A_0, A_1 \text{------} A_n = B$ of subsets of X such that $<A, B>$ is $\in$-d-chainable.

Theorem 4.2: Let $A,B \subset X$ and $<A,B>$ be d-chainable then $< \overline{A} , \overline{B} >$ is d-chainable.

Proof : Let $< A,B >$be d-chainable and $\epsilon^{/} > 0$ such that $2\epsilon^{/} < \epsilon$ Now $< A,B >$ is $\epsilon^{/}$-d-chainable $\Rightarrow \exists$a sequence.

$A = A_0, A_1 \text{------} A_n = B$ of subsets of X such that $A_{i-1} \subset V_{\epsilon^{//}}(A_i)$ and $A_i \subset V_{\epsilon^{//}}(A_{i-1})$

This implies

$$\overline{A_{i-1}} \subset \overline{V_{\epsilon'}}(A_i) \subset V_{2\epsilon'}(A_i) \subset \overline{V_\epsilon}(A_i)$$

and $$\overline{A_i} \subset \overline{V_{\epsilon''}}(A_{i-1}) \subset V_{2\epsilon''}(A_{i-1}) \subset \overline{V_\epsilon}(A_{i-1})$$

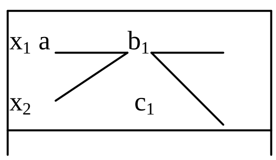
$\Rightarrow$ $(\overline{A}, \overline{B})$ is d chainable.

Theorem 4.3: Let $A,B \subset X$. If $(A \cup B)$ is connected and $\in > \max\{\delta(A), \delta(B)\}$ then $< A,B >$ is $\in$-d-chainable .

Proof : $A \cup B$ is connected $\Rightarrow A \cap \overline{B} \neq \phi$ or $\overline{A} \cap B \neq \phi$ suppose $A \cap \overline{B} \neq \phi$ then there exist an

$a \in A$ and $a \in \overline{B}$ .Now $a \in \overline{B} \Rightarrow a \in \bigcap_{\epsilon>0} V\in(B) \Rightarrow$ for each $\in>0$

there exist $b_1, c_1 \in B$ with $d(a, b_1, c_1) < \in$ Let $x_1, x_2 \in A$, $y_1, y_2 \in B$ then as $\in > \max\{\delta(A), \delta(B)\}$ and $x_1, x_2, a \in A \Rightarrow d(x_1, x_2, a) < \in$ hence x_1, x_2 is $\in$-d-chainable to b_1, c_1 in B.Again y_1, y_2 of B is $\in$-d-chainable to a. Hence by theorem 4.1 <A,B> is $\in$-d-chainable A similar result is obtained by taking $\overline{A} \cap B \neq \phi$



The following example shows that the convers part of the above theorem is not true for if (R, ρ) is the usual metric space and $(R\times R\times R, \rho)$ 2-metric space d is defined by

$$b = \text{Min}\{\rho(x,y), \rho(y,z), \rho(x,z)\}$$

$A = \{1/2, 1/3, \text{------------}\}$ and $B = \{1, 2/3\}$ subsets of R then

$\overline{A} = \{0, 1/2, 1/3, \text{------------}\}$ and $B = \overline{B}$ Also $\overline{A} \cap B = \phi$ and $A \cap \overline{B} = \phi \Rightarrow A \cup B$ is not connected $A \subset V\in(B)$ and $B \subset V\in(A) \Rightarrow < A, B >$ is $\in$-d-chainable where $\in = 1$.

Theorem 4.4 Let $F_1\ F_{2,}$--------- F_n ---------be a decreasing sequence of non empty compact sets in X and $F = \bigcap_{n\in N} F_n$ then there exits an $\in>0$ such that $<F, F_n>$ is $\in$-d-chainable $.\forall\ n \in N$

and $\Phi_\epsilon(<F, F_n>) = 1$

Proof: Let $\in>0$ be such that $\in > \delta(F_1)$ **Now** $F \subset F_1$ **Let** $x,y \in F_1$, $z \in F$

$\Rightarrow$ **$x,y,z \in F_1$ And $d(x,y,z) < \delta(F_1)$** $<\in \Rightarrow F_1 \subset V\in(F)$ Also $F \subset V\in(F_1)$

$\Rightarrow< F, F_1>$ **is** $\in$-d-chainable But $F_n \subset F_1 \subset V\in(F)$ $.\forall\ n \in N$ and $F \subset V\in(F_n)$ $\forall\ n \in N$

$\Rightarrow$ $<F, F_n>$ is $\in$-d-chainable $\forall\ n \in N$ and $\Phi_\epsilon(<F, F_n>) = 1$.

Theorem 4.5 Let F be the family of self $\in$- d-chainable subsets of X. Then the relation of **$\in$-d-chainability between two sets is an equivalence relation on F.**

proof (1) Reflexivity: Let $A \in F$ then by theorem 4.1 A is $\in$-d-chainable to A
Hence the relation of $\in$ -d-chainability is reflexive .

(2) Symmetric: Let $A,B \in F$ and A is $\in$-d-chainable to B then again by theorem 3.1 B is $\in$-d-chainable to A. Hence the relation of $\in$-d-chainability is symmetric

(3) Transitivity: Let A, B, C $\in$ F and let A is $\in$-d-chainable to B and B is $\in$-d-chainable to C then exists two sequence of subsets of F such that

$A = A_0, A_1, -----, A_{n1} = B$ ---- (1)

with $A_{i-1} \subset V_\in(A_i)$ and $A_i \subset V_\in(A_{i-1})$ $1 \le i \le n1$

Where n1 is the length of chain (1)

And

$B = B_0, B_1, -----B_{n2} = C$ --------(2)

with $B_{i-1} \subset V_\in(B_i)$ and $B_i \subset V_\in(B_{i-1})$ $1 \le i \le n2$

Where n2 is length of chain (2)

Clearly combination of these chains gives an $\in$-d-chain between A and C of Length n1+n2.hence the relation of $\in$ -d-chainability is transitive. Thus the relation of $\in$ -d-chainability between two subsets of F is an equivalence relation.

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